Factored Radix Numbers: A Quasi-Decimal Alternative to Higher Number Bases*

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Abstract

A novel non-standard positional numeral system is described, motivated by the need to represent numbers in a human-accessible format but using fewer digits than decimal would require. The factored radix numeral system FR(P,S) can represent a number with the same economy of digits as base PS but with the advantage, where $S = 10$, of easier transformation to and from decimal. A property of the system is that any FR(P,10) number differs from the decimal representation only if it uses fewer digits. For example, the decimal number `426` changes in base 20 to `116`, even though it still requires three digits, whereas the canonical FR(2,10) representation is `426`, as in decimal. Also described are two mnemonic higher-base digit alphabets, limited respectively by a seven-segment display constraint and the ASCII character set.

Keywords: numeral systems, systems of numeration, number representation, number systems, digit symbols.

1 Introduction

The general motivation for using a base higher than 10 to represent numbers is economy of representation. For example, the use of a base-20 system for Open Location Codes [1] produces shorter codes than would a decimal system. In many contexts, this benefit is secured at no cost, since the numbers are opaque to users. In other contexts, however, it can be desirable not only that a number be represented with fewer digits than decimal would require, but also that it be in a form convertible to decimal with relative ease by a person.

In an electronic context, consider a need to represent a number in the range 0…6000 on a display limited to three digits. A possible solution is to use a higher number base. In this case, base 19 would suffice, but it would be a poor choice, with place values of 1, 19, and 361 making the mental arithmetic for conversion to decimal unnecessarily difficult. Choosing a base that is a multiple of 10 improves this, but the sum-of-products-of-powers arithmetic remains difficult. Further, the values of the digits beyond the decimal range must be learned and remembered. Even for technical users, a display with more digits must strongly be preferred to a non-decimal numeral system.

A further constraint is possible. Imagine that a device with a three-digit display presents values in decimal, but later the range of values that it needs to display grows beyond the anticipated range. A software upgrade is required to represent values greater than 999 in three digits. To avoid confusion, any numeral which looks like decimal must continue to be safely interpretable as decimal, so a switch to base 20 which causes, e.g., 20 to be displayed as `10` is not acceptable.

Scenarios similar to these can be conceived with respect to physical objects which can accommodate only a limited number of digits, such as vehicle registration plates, or non-physical identifiers. To take a familiar series of numerals as an (unlikely) example of the latter, suppose that the RFC Oversight Committee wants to limit RFC numerals to four digits while preserving the ordinal values of the existing identifiers, since renumbering is impossible. What numeral is assigned to the RFC following RFC 9999? Y2K-style problems in general fit the theme.

In many contexts, it may be possible to sidestep the problem. For an electronic display, scrolling may be considered, or the digits of, e.g., a three-digit display could be flashed to indicate that 1000 should be added to the number shown, doubling the range; the decimal

*This document is https://mbreen.com/fr.pdf. To contact the author, see https://mbreen.com points of seven-segment displays, if otherwise unused, offer similar potential. In other cases, however, such workarounds may not be available (e.g., a flashing display may already carry a warning semantic). The rest of this article assumes a solution space restricted to the numeral system.

The following sections begin with a problem statement, followed by an examination of some alternative solutions. I then introduce the factored radix numeral system, examine its relative ease of conversion to decimal, and show how that may be improved with a mnemonic alphabet. I define its extension to fractional numbers, and note some inherent limitations. The factored radix system is then formally classified and compared with some other systems of numeration. The article concludes with a brief summary.

Terminology: In this article, a *digit* is any symbol that represents a number. A *decimal digit* is a digit in the decimal range (0..9), regardless of whether it appears in a decimal, vigesimal, or other number. A *numeral* is a string of one or more digits. A *numeral system* maps numerals to numbers (or *values*). A *decimal number* is the decimal representation of a number, which is not to be confused with a numeral which contains only decimal digits but is meant to be read as, e.g., vigesimal. Numerals are shown in a fixed-width font, e.g., `312`; when I want to refer to a number independent of its representation, it is in decimal in a standard font.

2 Problem statement

From the introduction, there are two distinct problems:

- *The interpretation problem*: Mental conversion of a numeral to the familiar decimal system can be difficult for two reasons: (a) *the mental arithmetic problem*, reflecting the number of steps and working memory required to perform any calculation that is required; (b) *the cipher problem*, reflecting any difficulty in learning and recalling the meaning of additional symbols.

- *The decimal superset problem*: Define a numeral system in which the decimal numerals have their decimal values but which extends the n-digit range beyond 10^n.

Any solution to the decimal superset problem is at least a partial solution also to the interpretation problem, since the first 10^n values require no interpretation. Therefore the latter problem can be a starting point.

0	`0123456789`
1	`abcdefghij`
2	`klmnopqrst`

Figure 1: Conventional alphabet of digits supporting numerals in bases up to base 30, arranged in decades.

3 Alternative decimal supersets

To define a superset of decimal is obviously not difficult. One possibility is to use decimal again, but with a different digit alphabet, e.g., using `a` for 0, `b` for 1, etc. If constrained to 4 digits then numbers 1. . . 9999 are in decimal after which `aaaa` is 10000, `aaab` is 10001, etc., up to `jjjj` for 19999; in general, the value of an n-digit numeral N in the second alphabet is $10^n + N_{10}$. Conversion of such a numeral to decimal involves no arithmetic: just replace each letter with the corresponding decimal digit and put a `1` in front. This can be called the *disjoint decimal* solution. If a doubling of the decimal range suffices then it is a good solution and nothing else needs to be considered.

Otherwise, to get the full 20^n range one would expect from 20 different digit symbols, an option is to use decimal up to 10^n, and then base 20 with a disjoint alphabet, e.g., `a`. . . `t` for 0. . . 19, for 10^n and larger numbers. In this system, `crnh` would have the value $2 \cdot 20^3 + 17 \cdot 20^2 + 13 \cdot 20 + 7 = 23067$. However, this avoids the mental arithmetic problem only for the first 10^n numbers.

For the rest of this article, every alphabet will begin with digits `0`. . . `9`. Until the subject of alphabets is considered again below, the usual convention will be followed of using `a` for digit value 10, `b` for 11, etc., as shown in Figure 1.

A variation on the above is to use the base 20 numerals generated by the alphabet of Figure 1 to extend the range, skipping any numeral composed only of decimal digits. Figure 2 shows this idea applied where the number of available digits $n = 4$. In this system, the value $v(N)$ of a numeral N containing a non-decimal digit is given by

$$v(N) = 10^n + N_{20} - \delta(N)$$

where N_{20} is the value of N considered as a base 20

numeral	value
`0`	0
...	
`9999`	9999
`000a`	10000
...	
`000j`	10009
`001a`	10010
...	
`jjjj`	159999

Figure 2: Decimal extended with non-decimal base 20 numerals.

number and $\delta(N)$ is the number of base 20 numbers less than N which have only decimal digits.

Noting that the largest base 20 number less than N which is composed only of decimal digits, *if considered as a decimal number*, is equal to a count of the numbers skipped excluding `0`, a little reflection shows that a general way to compute $\delta(N)$ is:

1. Find the most significant non-decimal digit in N.
2. Replace that and each less significant digit with `9`.
3. Treating the result as a decimal number, add 1.

As a complete example of conversion to decimal of a numeral with a non-decimal digit, consider $N = \texttt{2hg7}$:

$$v(\texttt{2hg7}) = 10^4 + \texttt{2hg7}_{20} - \delta(\texttt{2hg7})$$
$$\texttt{2hg7}_{20} = 2 \cdot 20^3 + 17 \cdot 20^2 + 16 \cdot 20 + 7 = 23127$$
$$\delta(\texttt{2hg7}) = 2999 + 1 = 3000$$
$$v(\texttt{2hg7}) = 10000 + 23127 - 3000 = 30127$$

The problem, of course, is that the procedure still involves base 20 conversion arithmetic. If the non-decimal base 20 numerals are to be used then they must be ordered according to a different principle.

4 The factored radix numeral system

The basis of the factored radix numeral system FR(P,S) is quickly seen by example. The decimal number `123456` may be converted to FR(2,10) as follows, illustrated in Figure 3:

`123456`	decimal
separated into:	
`1100`	prefix (`12`) expressed in base 2
`3456`	stem in base 10
combined into:	
`de56`	FR(2,10)

Figure 3: Conversion of a base 10 number to FR(2,10).

1. Lexically separate the numeral into a *prefix* and a *stem*. The prefix is constrained to be no longer than the stem when converted to base P: in this case, $P = 2$ and the first two digits (`12`) can be taken for the prefix.
2. Convert the prefix to base P (`1100`).
3. Take each digit in the stem and add to it the value of the corresponding digit in the base-P prefix multiplied by S. The result is the value of the corresponding digit in the FR(P,S) representation. In Figure 3, the first digit value is 13, which in the above alphabet is `d`.

The prefix will sometimes necessarily be empty, equivalent to a value of 0. Trivially, this is the case for all numbers less than S: the decimal number `7` is also `7` in FR(2,10), as it is in base 20. Unlike base 20, however, a decimal number such as `835` which cannot be shortened is unchanged when converted to FR(2,10) (whereas in base 20 it becomes `21f`).

A numeral in FR(P,S) is in canonical form if it is the minimum length representation, that is, if its prefix component is the maximum possible length. However, any shorter prefix can be chosen, including an empty one. This means that any decimal number is also a FR(P,10) number with the same value, for any P. In practice, a specified length will determine the representation of each number: for a four-digit display, it makes sense to use decimal up to `9999` and thereafter to use the minimum prefix required to limit the length to four digits; in effect, the maximum length is also the desired length.

Figure 4 shows a conversion in the reverse direction, this time from a numeral in FR(3,10) (the factored radix equivalent of base 30) to decimal. The steps are:

1. Divide each digit value by S (10): the quotient gives the corresponding digit of the base-P prefix, and the remainder gives the value of the corresponding digit of the stem. In this example, there is a digit `n` which has the value $23 = 2 \cdot 10 + 3$.

`05n9`	FR(3,10)
decomposed:	
`0020`	prefix in base 3 (decimal 6)
`0539`	stem in base 10
concatenated:	
`60539`	decimal

Figure 4: Conversion from FR(3,10) to base 10.

2. Convert the prefix to base S.

3. Concatenate the stem and the prefix.

The conversion procedures follow directly from the formal definition, which is more concise: Using $P\Pi S$ as a "pseudo-base" suffix, the value $N_{P\Pi S}$ of a numeral N in FR(P,S), $P > 1, S > 1$ of n digits with values $d_{n-1} \ldots d_0$, each in the range $0 \ldots PS-1$, is

$$N_{P\Pi S} = \sum_{i=0}^{n-1} (d_i \bmod S) S^i + S^n (\lfloor d_i / S \rfloor) P^i$$

It is easily seen that the range of an n-digit numeral in FR(P,S) is equal to the product of the range of the prefix and the range of the stem, and so is equal to the range of an n-digit numeral in base PS:

$$P^n S^n = (PS)^n$$

When FR(3,10) is described as being *equivalent* to base 30 this is what is meant.

5 Ease of interpretation

In respect of the mental arithmetic dimension, the interpretability of, for example, FR(2,10) relative to base 20 rests on the separation of the numeral into stem and prefix components. Just as, e.g., multiplication or division by a power of two becomes easier when a number is represented in binary, composing the decimal representation from the two parts is a trivial lexical operation with no mental arithmetic.

In the felicitous case, the decimal form of the number does not need to be shortened: there is no prefix and the stem is simply the decimal number unchanged. This compares well to the base 20 alternative where, unless it is a single digit, arithmetic is always required. Of course, in the case of, e.g., $n = 4$ (four digits), the felicitous case amounts to only $\frac{1}{16}$ of the range. However, it may still be that the numbers to be represented fall more frequently in that decimal range, so that, much of the time, no interpretation is necessary.

Where the factored radix form does differ, most of the digits of the decimal number will generally come from the stem (in the case of FR(2,10) with $n = 4$, four of at most six digits). Any decimal digit in the FR number is unchanged and in the same position in the stem. Each of the other digits of the stem is the value of the corresponding digit in the FR number modulo 10. For a person, the mental modulo-10 operation amounts to no more than choosing the units part of the decimal value.

Similarly, getting the digits of the base-P form of the prefix involves only choosing the tens part of each decimal value. Arithmetic is required only at the remaining step of converting the prefix from base P to base 10. Disregarding the lexical parts as comparatively insignificant, the effort in the arithmetic dimension therefore reduces to a conversion from base P to base 10. Mental conversion to decimal of an n-digit base 2 number is typically much easier than an n-digit base 20 number, and the same is true for $P > 2$.

While the arithmetic is much easier, it may be noted that there remains a potential conceptual gap unmentioned above: for readers of this article, the concept of numbers in a base other than 10 will be intimately familiar, but for others it may require explanation; otherwise, factored radix numbers will be transparent only in the decimal range.

The remaining effort is in the cipher problem, that is, memorizing the values of the non-decimal digits. This is a problem which affects FR(P,10) and base $10P$ equally. In the case of the latter, however, any easing of the cipher problem leaves a significant mental arithmetic hurdle. Where the mental arithmetic problem has already been addressed, the value in reducing the cipher problem is far greater. Thus, for factored radix numbers, it becomes very worthwhile to give some thought to the choice of alphabet.

6 Digit alphabets

The rule to construct a digit alphabet like that of Figure 1 is simple: append the letters of the Latin alphabet to the decimal digits; if necessary, use both uppercase and lowercase. While the rule to construct it is simple, one would not wish to need to do that to determine the value of a digit, or alternatively to do a mental count to recover its ordinal position. Any mnemonic basis for

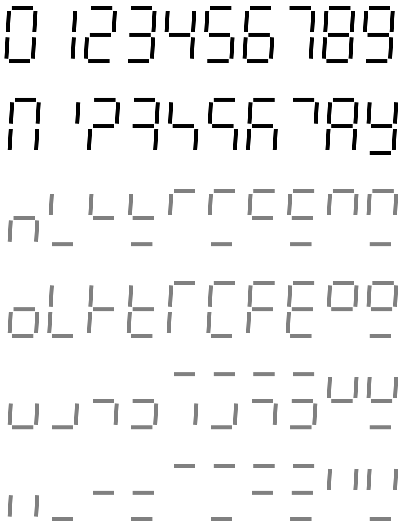

Figure 5: A mnemonic alphabet for a seven-segment display (up to base 60, with base 20 highlighted).

0	`0123456789`
1	`cjzwfsbvxq`
2	`nltmhgdrkp`
3	`CJZWFSBVXQ`
4	`NLTMHGDRKP`
5	`uiyeaUIYEA`

Figure 6: The Decimal Morphology Alphabet (DMA).

`2022`	decimal
`40`	prefix (`20`) in base 5
`22`	stem in base 10
`T2`	FR(5,10), DMA

Figure 7: Representing a year in FR(5,10).

learning and recall therefore has potential value. This section presents two such schemes, one suited to the constrained context of a seven-segment display, and another limited to the Latin alphabet. These are illustrative, not normative: the factored radix system is not defined by the choice of alphabet.

A seven-segment display is designed for decimal digits and cannot show certain letters, such as `W`. However, an arbitrary subset of its segments can be activated. Figure 5 shows a possible choice of alphabet. Each digit in the range 10. . . 19 is the same as the corresponding digit in the decimal range but with one segment switched off. A binary/octal principle is used for additional decades, supporting up to FR(6,10).

If the range is adequate, FR(2,10) is clearly the easiest choice: to get the value of the prefix in binary simply note which digits are "broken"; to get the digits of the stem, just copy the existing numbers, "fixing" any broken digits.

For an ASCII character set, Figure 6 shows an alphabet also based on a similarity principle, the Decimal Morphology Alphabet (DMA). Here, `z` (value 12) is chosen because it is shaped like `2`, and the modulo-10 correspondence for `s`, `b`, and `q` is equally direct. The letters `w`, `f`, `v` are the result of rotating the corresponding digit clockwise, while `x` is a clipped `8`. The lowercase digits for multiples of 10 are clipped `0`s, ascending in alphabetical order.

The "twenties" use weaker, second-choice mnemonics, where resemblance involves a rotation other than a quarter turn clockwise or a reflection. The digit `l` is deemed inferior to `j` in that it is confusable in some fonts with `1`. The digit `k` is a deformed `x`, and the mnemonic basis for digits `t` (22) and `g` (25) is sufficiently weak that they may be learned as exceptions. The last decade is included for completeness but is best avoided, both because using vowels introduces an increased likelihood of collisions with natural language words, and because some letters have lost any mnemonic correspondence to their modulo-10 values.

Once again, therefore, FR(2,10) with the base 20 subset is obviously easiest to use: to derive the `1`s of the prefix in binary, just note which digits are letters; to recover the decimal digits of the stem, just replace any letter with the standard decimal digit it resembles.

Where the range of FR(2,10) is inadequate, it is almost as easy to move to FR(5,10) as FR(3,10), since the thirties and forties are simply the uppercase versions of the preceding decades. Conversion of a base 5 prefix to decimal is also comparatively easy. Using FR(5,10), two digits can represent a value up to 2499, a range adequate for Gregorian years for some centuries to come. Figure 7 shows an example. Using the DMA, the 2020s are `T0...T9`, the 2030s are `M0...M9`, and so on.

Again, these alphabets merely demonstrate the potential ease of interpretation of the factored radix form. The choice of alphabet in a particular case may depend on a variety of context-specific constraints.

`4v.5`	FR(2,10), DMA.
`247.5`	decimal

Figure 8: A fractional FR(2,10) numeral and decimal equivalent.

7 Fractional numbers

For fractional numbers, it is easiest to begin by stating the semantics of a factored radix numeral with a decimal point (strictly, *radix separator*, which may be a comma; a point is used here). The simple rule is that the value in decimal (for $S = 10$) is found by converting the digits of the FR numeral as if it were a whole number, i.e., as if the decimal point were absent, but keeping the decimal point in the same position in the stem. Figure 8 shows an example.

All digits continue to be significant, including not only leading zeros but also any trailing zeros, despite the decimal point. In Figure 8, if `4v.5` were to be replaced with `4v.50` then the value in decimal would change from `247.5` to `447.5`.

Fractional numbers introduce additional considerations in the conversion to factored radix form. Is the decimal point to be in a fixed position or is it be moved to include more digits of a small number, or absent in the case of a large number? An appropriate unit scale must be chosen, bearing in mind that a number in which the integer part is zero will be unchanged in factored radix form (since the prefix cannot consume the decimal point). The decimal point makes it possible to drop less significant digits, but this must be done *before* conversion: for FR(2,10) DMA and a maximum width of four digits, not counting the decimal point, decimal `120.762` becomes `c.v62`, but `160.762` must be rounded to `160.76` before being converted to `60.7b`.

In short, the interpretation of a factored radix number which includes a decimal point remains facile, but procedures to construct it may vary.

8 Limitations

Numbers in factored radix form are clearly not designed to be amenable to native arithmetic. This should not be a concern, given that in many contexts mental arithmetic would not typically be required. In other scenarios it is likely to involve a conversion anyway, for example, where one wishes to know the decimal year 15 years after a year X expressed in factored radix form.

The construction of factored radix numerals also frustrates lexicographic sorting, regardless of the alphabet. Again, this should not be a concern, since manual sorting of numerals should be unusual.

Although not relevant to its use as descibed here, it is worth noting that the factored radix system produces numerals in which leading zeros are significant, but cannot itself shorten a numeral with a significant leading zero: a `0` prefix, if encoded, would be indistinguishable from an empty prefix.

9 Comparisons

The literature on numeral systems is extensive, ranging from “anormal systems” [2] (e.g., a system based on the golden ratio [3]) to Zeckendorf representation [4]. For this article, contributions can be divided into those concerned with the parameters of the familiar positional systems and those which differ from such systems in formal structure.

The standard integer-radix place-value system has invariably been taken for granted in modern proposals motivated by human factors. The concern of these contributions has instead been with the choice of base, e.g., advocating the general adoption of base 12, or with choosing alternative digit symbols with a more mnemonic or logical basis. Examples of the latter from the nineteenth and twentieth centuries respectively include Berdellé’s octal digits based on their binary representation, described in [5], and Lapointe’s hexadecimal digits based the idea of drawing a line to join a subset of four points [6]. In respect of alternative decimal alphabets, a recent proposal uses hollow digit symbols which allow for the concentric superposition of scaled digits within a square, similar to a QR code [7], while an idea from a century ago is an adaptation of tally mark aggregation as used in systems like Babylonian cuneiform [8]. Historically, another principle is seen in the Attic Greek acrophonic symbols where the first letter of the word for, e.g., “ten”, $\Delta\epsilon\kappa\alpha$, is used as the digit for that value [9]. The principle of Latin letters chosen for modulo-10 morphological similarity to the standard digits appears to be original to the author. It is not surprising that the problem of a mnemonic basis for base-X alphabets where X is greater than around 16 has not attracted much attention, given that general use by humans would entail memorizing a large multiplication table, without much advantage over a lower base such as 12.

Structurally, a factored radix numeral system FR(P, S) may be classified broadly as a non-standard positional numeral system: it is positional in that the contribution of each digit to a numeral's value depends on the position of the digit, but non-standard in that it also depends on how many digits there are.

Arithmetic has been a major driver of structural novelty in numeral systems, most obviously in the abandonment of Roman numerals for the decimal place-value system, and efficient arithmetic remains a key concern where it is performed by a machine. An example of another non-standard positional system, but one described with hardware implementation in mind, is quote notation [10]. This is effectively an extension of the radix complement concept combined with two repetition operators allowing any rational number to be represented in a finite number of digits. Its motivation is not only precision but the simplification of division in particular.

A contrasting non-positional system, the residue number system (RNS), represents an integer value by the remainders resulting from its division by a set of carefully chosen integers [11]. This modulo operation in the conversion to RNS is a minor element in common with factored radix numbers (where the stem is the decimal number modulo 10^n). As with factored radix numbers, the resulting representation frustrates sorting and some arithmetic: for division, numbers are converted back to standard positional form, and that reverse conversion itself is not straightforward [12]. Where the residue system is used, such compromises are accepted because it allows efficient native addition and multiplication of large integers, operations which can dominate the arithmetic of applications such as cryptography.

A numeral system with similar specialized application is the double-base number system [13]. This is nominally akin to the factored radix system in that it employs a kind of "two-dimensional" radix. However, it is based on the quite distinct principle of representing numbers as a sum of terms in the form $p^\alpha q^\beta$, where p and q are primes, e.g., with $p = 2, q = 3$, $127 = 2^2 3^3 + 2^1 3^2 + 2^0 3^0$. Again, this system and multi-base generalizations of it [14] are designed for computer arithmetic, not for human beings.

From the existing literature one might infer that any non-standard system can be suited only to machines, and that any worthwhile numeral system must support facile native arithmetic for at least a specialized subset of operations. The factored radix system is a novel counterexample, being specialized instead to representation in a constrained context, where it is designed to add to the economy of a higher number base a "decimal-compatible" facility of interpretation.

10 Conclusion

The factored radix numeral system FR(P, S) has been defined in a general way, but as a system specialized for its representational properties rather than for arithmetic, S is likely always to be 10 in practice and only P will vary. Where $P = 2$, which may suffice in many contexts, the interpretation of a factored radix number has been shown to be especially easy.

The utility of factored radix numbers, as motivated, is predicated on a hard constraint on the number of available digits for representing a number, or at least on a high cost to providing sufficient digits for a decimal representation, whether in a physical context or a logical one. Such a constraint can be expected rarely to occur in practice without the possibility of a workaround other than an alternative numeral system. Even where the solution must be in the numerals, the simple idea of using a disjoint decimal alphabet to merely double the range may suffice—in which case using the second decade of one of the alphabets suggested above could further improve that solution.

While the problem frame is narrow, within that frame the factored radix numeral system has a marked advantage over the alternative of representing values using a higher base, not only in the ease of interpretation of larger numbers afforded by its structural basis, but also in the simple property of being a value-preserving superset of decimal.

A Unix-style filter program to convert whole numbers between base S and FR(P, S) is available for reference and experimentation [15].

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